DETERMINANTS

16/08/2023

$$A = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12}\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13}\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

$$= a_{11}(a_{22}a_{33} - a_{32}a_{23}) - a_{12}(a_{21}a_{33} - a_{31}a_{23}) + a_{13}(a_{21}a_{32} - a_{31}a_{22})$$

- Properties of Δ :-

1. Rows $\rightarrow$ Col. & Col. $\rightarrow$ Rows $\Rightarrow$ $\Delta' = \Delta$

2. 2 Rows / Col: interchanged $\Rightarrow$ $\Delta' = -\Delta$

3. (2 Rows/Col. have same corresponding elems.) $\Rightarrow$ $\Delta = 0$

4. (Elems. of one row/col. multiplied by K) $\Rightarrow$ $\Delta' = K\Delta$

5. $$\begin{vmatrix} a_1 + A_1 & a_2 & a_3 \\ b_1 + B_1 & b_2 & b_3 \\ c_1 + C_1 & c_2 & c_3 \end{vmatrix} = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_2 \\ c_1 & c_2 & c_3 \end{vmatrix} + \begin{vmatrix} A_1 & a_2 & a_3 \\ B_1 & b_2 & b_3 \\ C_1 & c_2 & c_3 \end{vmatrix}$$

6. $$\begin{vmatrix} a_1 + la_2 + ma_3 & a_2 & a_3 \\ b_1 + lb_2 + mb_3 & b_2 & b_3 \\ c_1 + lc_2 + mc_3 & c_2 & c_3 \end{vmatrix} = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

7. (Elems. of one row/col. are all ZERO) $\Rightarrow$ $\Delta = 0$

8. If a det. $\Delta = 0$ for $x = a \Rightarrow (x-a) \mid \Delta$

Corollary : If two row/col. become identical for $x = a \Rightarrow (x-a) \mid \Delta$

General : If $(r+1)$ row/col. become identical for $x = a \Rightarrow (x-a)^r \mid \Delta$

- Elementary Transformations –

$$R_1 \longrightarrow R_1 + \alpha R_2 + \beta R_3$$

OR

$$C_1 \longrightarrow C_1 + \alpha C_2 + \beta C_3$$

Choose α, β as per convenience.

Q (i) If max. & min. value of $\begin{vmatrix} 1+s^2 & c^2 & s2x \\ s^2 & 1+c^2 & s2x \\ s^2 & c^2 & 1+s2x \end{vmatrix}$ are α & β, then prove that $\alpha^{2n} - \beta^{2n}$ is always an even integer

(ii) If $a, b, c > 0$ & $x, y, z \in R$, then find the value of $\begin{vmatrix} (a^x + a^{-x})^2 & (a^x - a^{-x})^2 & 1 \\ (b^y + b^{-y})^2 & (b^y - b^{-y})^2 & 1 \\ (c^z + c^{-z})^2 & (c^z - c^{-z})^2 & 1 \end{vmatrix}$.

(iii) If $\begin{vmatrix} x^n & x^{n+2} & x^{n+3} \\ y^n & y^{n+2} & y^{n+3} \\ z^n & z^{n+2} & z^{n+3} \end{vmatrix} = (y-z)(z-x)(x-y)\left(\frac{1}{x} + \frac{1}{y} + \frac{1}{z}\right)$, then find n

(iv) If $a \neq p$, $b \neq q$, $c \neq r$ & $\begin{vmatrix} p & b & c \\ a & q & c \\ a & b & r \end{vmatrix} = 0$,

then find $\frac{p}{p-a} + \frac{q}{q-b} + \frac{r}{r-c}$

(v) If A, B, C are the angles of ΔABC, then find the value of $\begin{vmatrix} e^{2iA} & e^{-iC} & e^{-iB} \\ e^{-iC} & e^{2iB} & e^{-iA} \\ e^{-iB} & e^{-iA} & e^{2iC} \end{vmatrix}$

(vi) For fixed $n \in \mathbb{N}$ if $\Delta = \begin{vmatrix} n! & (n+1)! & (n+2)! \\ (n+1)! & (n+2)! & (n+3)! \\ (n+2)! & (n+3)! & (n+4)! \end{vmatrix}$

then show that $\left[\frac{\Delta}{(n!)^3} - 4\right]$ is divisible by 'n'.

(vii) If A, B, C are the angles of a ΔABC

& $\begin{vmatrix} 1 & 1 & 1 \\ 1+\sin A & 1+\sin B & 1+\sin C \\ \sin A+\sin^2 A & \sin B+\sin^2 B & \sin C+\sin^2 C \end{vmatrix} = 0$,

then prove that ΔABC is an isosceles Δ.

A. (i) $R_1 \to R_1 - R_2$, $R_3 \to R_3 - R_2$ $\Rightarrow \begin{vmatrix} 1 & -1 & 0 \\ s^2 & 1+c^2 & s2x \\ 0 & -1 & 1 \end{vmatrix}$

$C_2 \to C_2 + C_1$ $\Rightarrow \begin{vmatrix} 1 & 0 & 0 \\ s^2 & 2 & s2x \\ 0 & -1 & 1 \end{vmatrix} = \underline{s2x + 2}$

$\alpha = 3$, $\beta = 1$ $\Rightarrow 3^{2n} - 1^{2n} = 9^n - 1 \Rightarrow 2 \mid 9^n - 1$

(Trivial)

(ii)
$$\begin{vmatrix} a^{2x}+a^{-2x}+2 & a^{2x}+a^{-2x}-2 & 1 \\ b^{2y}+b^{-2y}+2 & b^{2y}+b^{-2y}-2 & 1 \\ c^{2z}+c^{-2z}+2 & c^{2z}+c^{-2z}-2 & 1 \end{vmatrix}$$

$C_1 \to C_1 - C_2 \Rightarrow \begin{vmatrix} 4 & a^{2x}+a^{-2x}-2 & 1 \\ 4 & b^{2y}+b^{-2y}-2 & 1 \\ 4 & c^{2z}+c^{-2z}-2 & 1 \end{vmatrix} = 0$

(iii) $(xyz)^n \begin{vmatrix} 1 & x^2 & x^3 \\ 1 & y^2 & y^3 \\ 1 & z^2 & z^3 \end{vmatrix} = (xyz)^n (x-y)(y-z)(z-x)(xy+yz+zx)$

$\Downarrow$

$\underline{n=-1}$

(iv) $\begin{vmatrix} p & b & c \\ a & q & c \\ a & b & r \end{vmatrix} = 0 \Rightarrow p(qr-br-cq+bc)$
$+q(pr-pc-ar+ac)$
$+r(pq-pb-aq+ab)$
$= 2(p-a)(q-b)(r-c)$

$\Rightarrow \dfrac{p}{(p-a)} + \dfrac{q}{(q-b)} + \dfrac{r}{(r-c)} = 2$

(v) $e^{2iA}\big(\cancel{e^{2i\overbrace{(B+C)}^{\pi-A}} - e^{-2iA}}_{0}\big) + e^{-iC}\big(e^{-i\overbrace{(A+B)}^{\pi-C}} - e^{iC}\big) + e^{-iB}\big(e^{-i\overbrace{(A+C)}^{\pi-B}} - e^{iB}\big)$

$\Rightarrow (-2)\left[e^{-iC}e^{iC} + e^{-iB}e^{iB}\right] = -4$

(vi) $\Delta = (n!)^3(n+1)^2(n+2)\begin{vmatrix} 1 & 1 & 1 \\ n+1 & n+2 & n+3 \\ (n+1)(n+2) & (n+2)(n+3) & (n+3)(n+4) \end{vmatrix}$

$C_3 \to C_3 - C_2$
$C_2 \to C_2 - C_1 \Rightarrow \dfrac{\Delta}{(n!)^3} = (n+1)^2(n+2)\begin{vmatrix} 1 & 0 & 0 \\ (n+1) & 1 & 1 \\ (n+1)(n+2) & 2(n+2) & 2(n+3) \end{vmatrix}$

$$\Rightarrow \quad \frac{\Delta}{(n!)^3} - 4 = 2\left[\underbrace{(n+1)^2(n+2) - 2}_{\text{const. term} = 0}\right]$$

$$\Rightarrow \quad \boxed{n \;\Big|\; \left[\frac{\Delta}{(n!)^3} - 4\right]}$$

$\square$

7. $$\begin{vmatrix} 1 & 1 & 1 \\ 1+s_A & 1+s_B & 1+s_C \\ s_A+s_A^2 & s_B+s_B^2 & s_C+s_C^2 \end{vmatrix} = \cancel{\begin{vmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ s_A+s_A^2 & s_B+s_B^2 & s_C+s_C^2 \end{vmatrix}}^{\,0} + \begin{vmatrix} 1 & 1 & 1 \\ s_A & s_B & s_C \\ s_A+s_A^2 & s_B+s_B^2 & s_C+s_C^2 \end{vmatrix}$$

$$= \cancel{\begin{vmatrix} 1 & 1 & 1 \\ s_A & s_B & s_C \\ s_A & s_B & s_C \end{vmatrix}}^{\,0} + \begin{vmatrix} 1 & 1 & 1 \\ s_A & s_B & s_C \\ s_A^2 & s_B^2 & s_C^2 \end{vmatrix}$$

$$\Delta = (s_A - s_B)(s_B - s_C)(s_C - s_A)$$

Given $\Delta = 0 \Rightarrow$ $s_A = s_B$ or $s_B = s_C$ or $s_C = s_A$

$\because$ $A, B, C \neq 0$

$\Rightarrow$ At least 2 of A, B, C are identical

$\therefore$ ΔABC is isosceles.

→ Operations on Det

- Product

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} \times \begin{vmatrix} \alpha_1 & \beta_1 & \gamma_1 \\ \alpha_2 & \beta_2 & \gamma_2 \\ \alpha_3 & \beta_3 & \gamma_3 \end{vmatrix}$$

$$= \begin{vmatrix} a_1\alpha_1 + b_1\beta_1 + c_1\gamma_1 & a_1\alpha_2 + b_1\beta_2 + c_1\gamma_2 & a_1\alpha_3 + b_1\beta_3 + c_1\gamma_3 \\ a_2\alpha_1 + b_2\beta_1 + c_2\gamma_1 & a_2\alpha_2 + b_2\beta_2 + c_2\gamma_2 & a_2\alpha_3 + b_2\beta_3 + c_2\gamma_3 \\ a_3\alpha_1 + b_3\beta_1 + c_3\gamma_1 & a_3\alpha_2 + b_3\beta_2 + c_3\gamma_2 & a_3\alpha_3 + b_3\beta_3 + c_3\gamma_3 \end{vmatrix}$$

NOTE: Multiplication can also be carried out like matrices (i.e row by col.)

- Differentiation

$$\Delta(x) = |C_1\ C_2\ C_3| \Rightarrow \Delta'(x) = |C_1'C_2C_3| + |C_1C_2'C_3| + |C_1C_2C_3'|$$

$$\Delta(x) = \begin{vmatrix} R_1 \\ R_2 \\ R_3 \end{vmatrix} \Rightarrow \Delta'(x) = \begin{vmatrix} R_1' \\ R_2 \\ R_3 \end{vmatrix} + \begin{vmatrix} R_1 \\ R_2' \\ R_3 \end{vmatrix} + \begin{vmatrix} R_1 \\ R_2 \\ R_3' \end{vmatrix}$$

- Summation

If $\Delta(r) = \begin{vmatrix} f(r) & a & l \\ g(r) & b & m \\ h(r) & c & n \end{vmatrix}$

$$\Rightarrow \sum_{r=1}^{n} \Delta(r) = \begin{vmatrix} \sum_{r=1}^{n} f(r) & a & l \\ \sum_{r=1}^{n} g(r) & b & m \\ \sum_{r=1}^{n} h(r) & c & n \end{vmatrix}$$

Same holds true if only one row dependant on 'r'

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• Integration

$$\int \Delta(x) = \begin{vmatrix} \int f(x) & a & l \\ \int g(x) & b & m \\ \int h(x) & c & n \end{vmatrix}$$

Q. (viii) P.T $\begin{vmatrix} \sin(x+\alpha) & \cos(x+\alpha) & \sin\alpha \\ \sin(x+\beta) & \cos(x+\beta) & \sin\beta \\ \sin(x+\gamma) & \cos(x+\gamma) & \sin\gamma \end{vmatrix}$

is independent of x.

(ix) In ΔABC, P.T $\begin{vmatrix} \sin 2A & \sin C & \sin B \\ \sin C & \sin 2B & \sin A \\ \sin B & \sin A & \sin 2C \end{vmatrix} = 0$

A (viii) $\Delta'(x) = \begin{vmatrix} \cos(x+\alpha) & \cos(x+\alpha) & \sin\alpha \\ \cos(x+\beta) & \cos(x+\beta) & \sin\beta \\ \cos(x+\gamma) & \cos(x+\gamma) & \sin\gamma \end{vmatrix} + \begin{vmatrix} \sin(x+\alpha) & -\sin(x+\alpha) & \sin\alpha \\ \sin(x+\beta) & -\sin(x+\beta) & \sin\beta \\ \sin(x+\gamma) & -\sin(x+\gamma) & \sin\gamma \end{vmatrix}$

$+ \begin{vmatrix} \sin(x+\alpha) & \cos(x+\alpha) & 0 \\ \sin(x+\beta) & \cos(x+\beta) & 0 \\ \sin(x+\gamma) & \cos(x+\gamma) & 0 \end{vmatrix}$

$= 0$

$\Delta'(x) = 0 \Rightarrow \Delta(x) = \text{const.} \Rightarrow$ Indep. of x.

(ix)
$$\begin{vmatrix} \Delta_A C_A + C_A \Delta_A + 0\cdot0 & \Delta_A C_B + C_A \Delta_B + 0\cdot0 & \Delta_A C_C + C_A \Delta_C + 0\cdot0 \\ \Delta_A C_B + C_B \Delta_A + 0\cdot0 & \Delta_B C_B + C_B \Delta_B + 0\cdot0 & \Delta_B C_C + \Delta_C C_B + 0\cdot0 \\ \Delta_A C_C + C_A \Delta_C + 0\cdot0 & \Delta_B C_C + C_B \Delta_C + 0\cdot0 & \Delta_C C_C + C_C \Delta_C + 0\cdot0 \end{vmatrix}$$

$$= \begin{vmatrix} \Delta_A & C_A & 0 \\ \Delta_B & C_B & 0 \\ \Delta_C & C_C & 0 \end{vmatrix} \times \begin{vmatrix} C_A & C_B & C_C \\ \Delta_A & \Delta_B & \Delta_C \\ 0 & 0 & 0 \end{vmatrix} = 0 \times 0 = 0$$

Q. (x) If $(x_1-x_2)^2+(y_1-y_2)^2 = a^2$, $(x_2-x_3)^2+(y_2-y_3)^2 = b^2$, $(x_3-x_1)^2+(y_3-y_1)^2 = c^2$, P.T

$$\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}^2 = \left(\frac{1}{4}\right) (a+b+c)(b+c-a)(c+a-b)(a+b-c)$$

(xi) If $ax_1^2+by_1^2+cz_1^2 = ax_2^2+by_2^2+cz_2^2 = ax_3^2+by_3^2+cz_3^2 = d$

& $ax_1x_3+by_1y_3+cz_1z_3 = ax_2x_3+by_2y_3+cz_2z_3 = ax_1x_2+by_1y_2+cz_1z_2 =$

P.T.
$$\begin{vmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{vmatrix} = (d-f)\left(\frac{a+2f}{abc}\right)^{1/2}$$

(xii) Let $a, b, c \in \mathbb{R}$ with $a^2+b^2+c^2=1$.

Show that the eqn

$$\begin{vmatrix} ax-by-c & bx+ay & cx+a \\ bx+ay & -ax+by-c & cy+b \\ cx+a & cy+b & -ax-by+c \end{vmatrix} = 0$$

represents a straight line

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(xiii) If $f(\theta) = \begin{vmatrix} \sec^2(\theta) & 1 & 1 \\ \cos^2(\theta) & \cos^2(\theta) & \csc^2(\theta) \\ 1 & \cos^2(\theta) & \cot^2(\theta) \end{vmatrix}$,

P.T

$$\int_0^{\pi/4} f(\theta)\, d\theta = \frac{3\pi + 8}{32}$$

(xiv) P.T

$$\begin{vmatrix} bc-a^2 & ca-b^2 & ab-c^2 \\ ca-b^2 & ab-c^2 & bc-a^2 \\ ab-c^2 & bc-a^2 & ca-b^2 \end{vmatrix} = \begin{vmatrix} r^2 & u^2 & u^2 \\ u^2 & r^2 & u^2 \\ u^2 & u^2 & r^2 \end{vmatrix}$$

where $r^2 = a^2+b^2+c^2$ & $u^2 = ab+bc+ca$

(xv) If $D_1 = \begin{vmatrix} a & b & c \\ d & e & f \\ g & h & k \end{vmatrix}$, $D_2 = \begin{vmatrix} a & g & x \\ b & h & y \\ c & k & z \end{vmatrix}$

& $d = tx$, $e = ty$, $f = tz$, P.T w/o expanding that $D_1 = -tD_2$

f) A. (x) LHS $= 4 \times (ar(\Delta))^2$

RHS $= 4 \left(\frac{\Sigma a}{2}\right)\left(\frac{\Sigma a}{2} - a\right)\left(\frac{\Sigma a}{2} - b\right)\left(\frac{\Sigma a}{2} - c\right)$

$= 4 \left(\sqrt{s(s-a)(s-b)(s-c)}\right)^2$

$= 4 \times (ar(\Delta))^2$

$\Rightarrow$ LHS = RHS

(xi) $\begin{vmatrix} ax_1 & by_1 & cz_1 \\ ax_2 & by_2 & cz_2 \\ ax_3 & by_3 & cz_3 \end{vmatrix} \begin{vmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \\ z_1 & z_2 & z_3 \end{vmatrix} = \begin{vmatrix} d & f & f \\ f & d & f \\ f & f & d \end{vmatrix}$

$\Rightarrow (abc)(\Delta)^2 = d(d^2-f^2) + f(f^2-df) + f(f^2-df)$

$= d(d+f)(d-f) + f^2(f-d) + f^2(f-d)$

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$$\Rightarrow \Delta = (d-f)\left(\frac{(d+2f)}{abc}\right)^{1/2}$$

(xii) $\begin{vmatrix} ax-by-c & bx+y & cx+a \\ bx+ay & -ax+by-c & cy+b \\ cx+a & cy+b & -ax-by+c \end{vmatrix}$

$$= \left(\frac{1}{abc}\right)\begin{vmatrix} a^2x-aby-ac & abx+ay & acx+a^2 \\ b^2x+aby & -abx+b^2y-bc & bcy+b^2 \\ c^2x+ac & c^2y+bc & -acx-bcy+c^2 \end{vmatrix}$$

$R_1 \to R_1+R_2+R_3 \Rightarrow \left(\frac{1}{abc}\right)\begin{vmatrix} x & y & 1 \\ b^2x+aby & -abx+b^2y+c & bcy+b^2 \\ c^2x+ac & c^2y+bc & -acx-bcy+c^2 \end{vmatrix}$

$[\because a^2+b^2+c^2=1]$

$R_3 \to a^2R_1+R_2+R_3 \Rightarrow = \left(\frac{1}{ac}\right)\begin{vmatrix} x & y & 1 \\ bx+ay & -ax+by-c & cy+b \\ x+aby+ac & y-abx & 1-acx \end{vmatrix}$

$R_3 \to R_3-R_1 \Rightarrow \left(\frac{1}{c}\right)\begin{vmatrix} x & y & 1 \\ bx+ay & -ax+by-c & cy+b \\ by+c & -bx & -cx \end{vmatrix}$

$R_2 \to R_2-bR_1 \Rightarrow \left(\frac{1}{c}\right)\begin{vmatrix} x & y & 1 \\ ay & -ax-c & cy \\ by+c & -bx & -cx \end{vmatrix}$

$$\Rightarrow \left(\frac{1}{cxy}\right)\begin{vmatrix} x^2 & y^2 & 1 \\ axy & -axy-cy & cy \\ bxy+cx & -bxy & -cx \end{vmatrix}$$

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$C_2 \to C_1 + C_2 + C_3$

$$\Rightarrow \left(\frac{1}{cxy}\right)\begin{vmatrix} x^2 & x^2+y^2+1 & 1 \\ axy & 0 & cy \\ bxy+cx & 0 & -cx \end{vmatrix} = 0$$

$$\Rightarrow \underbrace{(x^2+y^2+1)}_{\neq 0}(ax+by+c) = 0$$

$$\Rightarrow \boxed{ax+by+c=0}$$

□

(xiii) $R_2 \to R_2 - R_3$

$$\Rightarrow \begin{vmatrix} \sec^2(\theta) & 1 & 1 \\ -\sin^2(\theta) & 0 & 1 \\ 1 & \cos^2(\theta) & \cot^2(\theta) \end{vmatrix}$$

$$\Rightarrow \sec^2(\theta)\begin{vmatrix} 1 & c_\theta^2 & c_\theta^2 \\ -s_\theta^2 & 0 & 1 \\ 1 & c_\theta^2 & \cot^2(\theta) \end{vmatrix} = \sec^2(\theta)\begin{vmatrix} 1 & c_\theta^2 & c_\theta^2 \\ -s_\theta^2 & 0 & 1 \\ 0 & 0 & \cot^2(\theta)-c_\theta^2 \end{vmatrix}$$

$R_3 \to R_3 - R_1$

$$= \sec^2(\theta)\, c_\theta^2\, s_\theta^2\,(\cot^2(\theta) - c_\theta^2) = c_\theta^2 - s_\theta^2 c_\theta^2$$

$$f(\theta) = \frac{3}{8} + \frac{c_{2\theta}}{2} + \frac{c_{4\theta}}{8}$$

$$\int_0^{\pi/4} f(\theta)\, d\theta = \frac{3\pi+8}{32}$$

□

(xiv)
$$\begin{vmatrix} bc-a^2 & ca-b^2 & ab-c^2 \\ ca-b^2 & ab-c^2 & bc-a^2 \\ ab-c^2 & bc-a^2 & ca-b^2 \end{vmatrix} = \begin{vmatrix} bc-a^2 & ca-b^2 & ab-c^2 \\ (\Sigma a)(a-b) & (\Sigma a)(b-c) & (\Sigma a)(c-a) \\ (\Sigma a)(a-c) & (\Sigma a)(b-a) & (\Sigma a)(c-b) \end{vmatrix}$$

$R_2 \to R_2 - R_1$

$R_3 \to R_3 - R_1$

$$= (\Sigma a)^2 \begin{vmatrix} bc-a^2 & ca-b^2 & ab-c^2 \\ (a-b) & (b-c) & (c-a) \\ (a-c) & (b-a) & (c-b) \end{vmatrix} = (\Sigma a)^2 \begin{vmatrix} bc-a^2 & ca-b^2 & \Sigma ab - \Sigma a^2 \\ a-b & b-c & 0 \\ a-c & b-a & 0 \end{vmatrix}$$

$C_3 = C_1 + C_2 + C_3$

$$= (\Sigma a)^2(\Sigma ab - \Sigma a^2)\left((a-b)(b-a) - (a-c)(b-c)\right)^2$$

$$= \underline{(\Sigma a)^2(\Sigma ab - \Sigma a^2)^2}$$

$$\begin{vmatrix} r^2 & u^2 & u^2 \\ u^2 & r^2 & u^2 \\ u^2 & u^2 & r^2 \end{vmatrix} = \begin{vmatrix} r^2 & u^2 & u^2 \\ u^2-r^2 & r^2-u^2 & 0 \\ u^2-r^2 & 0 & r^2-u^2 \end{vmatrix} = \begin{vmatrix} r^2+2u^2 & u^2 & u^2 \\ 0 & r^2-u^2 & 0 \\ 0 & 0 & r^2-u^2 \end{vmatrix}$$

$R_2 \to R_2 - R_1$ & $R_3 \to R_3 - R_1$ $\quad$ $C_1 \to C_1 + C_2 + C_3$

$$= (r^2+2u^2)\,(r^2-u^2)^2$$

$$= \underline{(\Sigma a)^2(\Sigma a^2 - \Sigma ab)^2}$$

(xv) $$D_2 = \begin{vmatrix} a & g & x \\ b & h & y \\ c & k & z \end{vmatrix} = \begin{vmatrix} a & b & c \\ g & h & k \\ x & y & z \end{vmatrix} = -\begin{vmatrix} a & b & c \\ x & y & z \\ g & h & k \end{vmatrix}$$

$$\Rightarrow tD_2 = -\begin{vmatrix} a & b & c \\ tx & ty & tz \\ g & h & k \end{vmatrix} = -\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & k \end{vmatrix}$$

$$\Rightarrow \underline{D_1 = -tD_2}$$

$\square$

LINEAR SYSTEM OF EQ^{n}s

$$a_1x + b_1y + c_1z = d_1$$
$$a_2x + b_2y + c_2z = d_2$$
$$a_3x + b_3y + c_3z = d_3$$

In Matrix form,

$$\begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix}$$

Let $D = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$, $D_1 = \begin{vmatrix} d_1 & b_1 & c_1 \\ d_2 & b_2 & c_2 \\ d_3 & b_3 & c_3 \end{vmatrix}$

$D_2 = \begin{vmatrix} a_1 & d_1 & c_1 \\ a_2 & d_2 & c_2 \\ a_3 & d_3 & c_3 \end{vmatrix}$, $D_3 = \begin{vmatrix} a_1 & b_1 & d_1 \\ a_2 & b_2 & d_2 \\ a_3 & b_3 & d_3 \end{vmatrix}$

The following cases can arise :-

(I) $D \neq 0 \Rightarrow$ unique sol^n

Cramer's Rule : $\boxed{x = \frac{D_1}{D}}$, $\boxed{y = \frac{D_2}{D}}$, $\boxed{z = \frac{D_3}{D}}$

(II) $D = 0$ & at least one of $D_1, D_2, D_3 \neq 0$

$\Rightarrow$ Inconsistent sol^n

so, no sol^n

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(III) $D = 0$ & $D_1 = D_2 = D_3 = 0$

$\Rightarrow$ ∞ $sol^n s$ (when they are dependent)

OR no sol^n (when they are independent)

Q. Find the values of λ & μ for which

$$x+y+z=3$$
$$x+3y+2z=6$$
$$x+\lambda y+3z=\mu$$

has
(i) unique sol^n
(ii) no sol^n
(iii) ∞ $sol^n s$

A. (i) $D \neq 0 \Rightarrow \begin{vmatrix} 1 & 1 & 1 \\ 1 & 3 & 2 \\ 1 & \lambda & 3 \end{vmatrix} = \begin{vmatrix} 1 & 0 & 0 \\ 1 & 2 & 1 \\ 1 & (\lambda-1) & 2 \end{vmatrix}$

$\Rightarrow 4-(\lambda-1) \neq 0 \Rightarrow \underline{\lambda \neq 5}, \underline{\mu \in R}$

(ii) $D=0 \Rightarrow \underline{\lambda=5}$

1. $D_1 = 0 \Rightarrow \begin{vmatrix} 3 & 1 & 1 \\ 6 & 3 & 2 \\ \mu & 5 & 3 \end{vmatrix} = \begin{vmatrix} 3 & 1 & 1 \\ 0 & 1 & 0 \\ \mu & 5 & 3 \end{vmatrix} = 0$

$\Rightarrow 9-\mu=0 \Rightarrow \underline{\mu=9}$

2. $D_2=0 \Rightarrow \begin{vmatrix} 1 & 3 & 1 \\ 1 & 6 & 2 \\ 1 & \mu & 3 \end{vmatrix} = \begin{vmatrix} 1 & 3 & 1 \\ -1 & 0 & 0 \\ 1 & \mu & 3 \end{vmatrix} = 0$

$\Rightarrow 9-\mu=0 \Rightarrow \underline{\mu=9}$

3. $D_3=0 \Rightarrow \begin{vmatrix} 1 & 1 & 3 \\ 1 & 3 & 6 \\ 1 & 5 & \mu \end{vmatrix} = \begin{vmatrix} 1 & 1 & 3 \\ 0 & 2 & 3 \\ 0 & 4 & (\mu-3) \end{vmatrix} = 0$

$\Rightarrow \underline{\mu=9}$

MATRICES

→ Defn

- Row matrix — $\begin{bmatrix} a_1 & a_2 & a_3 \end{bmatrix}$

- Col. matrix — $\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}$

- Sq. matrix — # Row = # Col.

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

- Diagonal matrix — Sq. matrix s.t

$i = j \iff a_{ij} \neq 0$

$i \neq j \iff a_{ij} = 0$

$$\begin{bmatrix} a_{11} & 0 & 0 \\ 0 & a_{22} & 0 \\ 0 & 0 & a_{33} \end{bmatrix}$$

- Scalar matrix — Diag. matrix s.t all diagonal entries equal.

$$\begin{bmatrix} a & 0 & 0 \\ 0 & a & 0 \\ 0 & 0 & a \end{bmatrix}$$

- Identity (or unit) Matrix (I) — $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

- Upper Δ Matrix – Sq. matrix s.t

$i > j \Leftrightarrow a_{ij} = 0$

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{22} & a_{23} \\ 0 & 0 & a_{33} \end{bmatrix}$$

- Lower Δ matrix – Sq. matrix s.t

$i < j \Leftrightarrow a_{ij} = 0$

$$\begin{bmatrix} a_{11} & 0 & 0 \\ a_{21} & a_{22} & 0 \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

- Null matrix – $a_{ij} = 0 \;\; \forall \; i, j$

- Trace of a Matrix – (Def. only for sq. matrix)

$$\boxed{tr(A) = \left(\sum a_{ii}\right)}$$

(Sum of elem. of principal diagonal)

- Equality of Matrices –

① Matrices of same order

② $a_{ij} = b_{ij} \;\; \forall \; i, j$

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→ Operations

- Add^n — $A + B = C$

$$\Updownarrow$$

$$a_{ij} + b_{ij} = c_{ij}$$

- Multiplication —

Cond^n : $A \times B$ defined iff $n = p$

$A \to m \times n$
$B \to p \times q$
$\Downarrow$
$C = A \times B \to m \times q$

$$A \times B = \underset{3\times3}{\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}} \underset{3\times2}{\begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \\ b_{31} & b_{33} \end{bmatrix}}$$

$$= \underset{3\times2}{\begin{bmatrix} a_{11}b_{11} + a_{12}b_{21} + a_{13}b_{31} & a_{11}b_{12} + a_{12}b_{22} + a_{13}b_{33} \\ a_{21}b_{11} + a_{22}b_{21} + a_{23}b_{31} & a_{21}b_{12} + a_{22}b_{22} + a_{23}b_{33} \\ a_{31}a_{11} + a_{32}b_{21} + a_{33}b_{31} & a_{31}b_{12} + a_{32}b_{22} + a_{33}b_{33} \end{bmatrix}}$$

→ Properties

1. $AB \neq BA$
2. $(AB)C = A(BC)$
3. $A(B+C) = AB + AC$
4. $AI = IA = A$
5. $AB = 0 \not\Rightarrow A = 0$ or $B = 0$ or $A = B = 0$

→ Transpose

If $A_{n\times m}$ & $B_{n\times m}$ are matrices s.t

$$a_{ij} = b_{ji}$$

then B is called transpose of A

$$B = A^T \text{ or } A'$$

• Ppts —

1. $(A^T)^T = A$
2. $(A+B)^T = A^T + B^T$
3. $(kA)^T = k(A^T)$
4. $(AB)^T = B^T A^T$

• Symmetric / Skew-Symmetric Matrix —

Condⁿ : Only for sq. matrices

Symmetric — $A^T = A \Leftrightarrow a_{ij} = a_{ji}$

Skew-Symmetric — $A + A^T = 0 \Leftrightarrow a_{ij} + a_{ji} = 0$

* In skew-sym. mat., $a_{ii} + a_{ii} = 0 \Rightarrow a_{ii} = 0$

∴ all diagonal elem. of a skew-sym. mat. are ZERO.

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* Every sq. matrix can be uniquely expressed as a sum of sym. & skew-sym. mat.

$$A = \underbrace{\frac{1}{2}(A + A^T)}_{\text{Sym.}} + \underbrace{\frac{1}{2}(A - A^T)}_{\text{Skew-sym.}}$$

- Orthogonal matrix – Def. only for sq. mat.

A is called orthogonal mat. iff.

$$\boxed{AA^T = A^TA = I}$$

- Singular / Non-singular Matrix – (Def. only for sq. mat)

Singular – $|A| = 0$
Non-singular – $|A| \neq 0$

- Adjoint Matrix – (Def only for sq. mat.)

Transpose of Co-factor matrix.

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \qquad \left(\begin{array}{c}\text{Co-factor} \\ \text{mat. of } A\end{array}\right) = \begin{bmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{bmatrix}$$

C

$$C_{ij} = (-1)^{(i+j)} \det\left(\begin{array}{l}\text{Mat. formed by removing} \\ \text{row \& col. containing } a_{ij}\end{array}\right)$$

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eg $C_{11} = \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix}$ $C_{12} = - \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix}$

Hence, $adj(A) = C^T$

→ Inverse

Cond^n: Non-singular sq. mat.

$\forall$ A $\exists$ A^{-1} s.t $\boxed{AA^{-1} = A^{-1}A = I}$

$$\boxed{A = \frac{1}{|A|} adj(A)}$$

• Ppts

1. Inverse of a matrix is always unique

2. $\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{(ad-bc)} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$

3. $\begin{bmatrix} a_1 & 0 & \cdots \\ 0 & a_2 & \\ \vdots & & \\ & & a_n \end{bmatrix}^{-1} = \begin{bmatrix} \frac{1}{a_1} & 0 & \cdots \\ 0 & \frac{1}{a_2} & \\ \vdots & & \\ & & \frac{1}{a_n} \end{bmatrix}$; $a_i = 0$ $\forall\ i = 1, 2, \ldots n$

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4. $(AB)^{-1} = B^{-1}A^{-1}$
5. $(A^T)^{-1} = (A^{-1})^T$

- Ppts. of adj

1. $(adj(A))A = A(adj(A)) = |A|I$

2. if A is singular $\Leftrightarrow$ $(adj(A))A = A(adj(A)) = O$ - (null matrix)

3. $|adj(A)| = |A|^{(n-1)}$

Proof: $|adj(A)(A)| = ||A|I| \Rightarrow |adj(A)|\,|A| = |A|^n$
$\Rightarrow |adj(A)| = |A|^{(n-1)}$

4. $adj(AB) = adj(B)\,adj(A)$

5. $(adj(A))^T = adj(A^T)$

☆ 6. $adj(adj(A)) = |A|^{(n-2)}A$

7. adj (diagonal mat) is a diago mat.

* $|AB| = |A|\,|B|$

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Q. (i) If $A = \begin{bmatrix} c\theta & s\theta \\ -s\theta & c\theta \end{bmatrix}$ & $A \, adj(A) = \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, then find λ

(ii) If $A = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$ & $A^2 - 4A - nI = 0$, find n.

(iii) If for matrices A & B, $AB = A$ & $BA = B$, then prove that $A^2 = A$

(iv) If $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$, then prove that $A^n = \begin{bmatrix} 1 & n \\ 0 & 1 \end{bmatrix}$

A. (i) $A \, adj(A) = |A| \Rightarrow \lambda = c^2 + s^2 = \underline{1}$

(ii) $A^2 = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} 5 & -4 \\ -4 & 5 \end{bmatrix}$

$A^2 - 4A - nI = \begin{bmatrix} -3-n & 0 \\ 0 & -3-n \end{bmatrix} \Rightarrow \underline{n = -3}$

(iii) $(AB)(A) = A \cdot A \Rightarrow A(BA) = A^2 \Rightarrow AB = A^2$
$\Rightarrow \underline{A = A^2}$

(iv) Base step : $A^2 = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$

Induction step : $A^n = A \cdot A^{(n-1)} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & (n-1) \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & n \\ 0 & 1 \end{bmatrix}$

$\therefore$ Result follows $\forall \, n \in \mathbb{N}$